

3D Viewing Episode 2

1 Positioning and Orienting the Camera

- Recall that our projection calculations, whether orthographic or frustum/perspective, were made with the camera at $(0, 0, 0)$ looking down the $-z$ axis.
- The camera is also vertically oriented — i.e., \vec{up} is the vector $\langle 0, 1, 0 \rangle$.
- But we want the camera to be anywhere, looking anywhere; how is this done?
- Recall that transforming an object is equivalent to transforming its axes in the opposite manner: e.g. moving an object centered on the origin to the right by 10 units is visually identical to moving the origin (and its axes) to the *left* by 10 units.
- Suppose that we would like to position our camera at some point P — the *center of projection* (COP). The camera is looking toward Q — the *look-at* or *eye* point — and is oriented according to the \vec{up} vector, which indicates what is vertical relative to the camera. Figure 1 illustrates this setup.

2 Deriving the Transformation

- The idea here is to derive the transformation such that $\langle 1, 0, 0 \rangle \rightarrow \vec{x}_e$, $\langle 0, 1, 0 \rangle \rightarrow \vec{y}_e$, and $\langle 0, 0, 1 \rangle \rightarrow \vec{z}_e$.
- \vec{x}_e , \vec{y}_e , and \vec{z}_e are vectors, so they can be broken down in terms of their components as $\langle x_{e_x}, x_{e_y}, x_{e_z} \rangle$, $\langle y_{e_x}, y_{e_y}, y_{e_z} \rangle$, and $\langle z_{e_x}, z_{e_y}, z_{e_z} \rangle$.
- Thus, the matrix that transforms the standard cartesian axes into axes relative to the camera is:

$$\begin{bmatrix} x_{e_x} & y_{e_x} & z_{e_x} & 0 \\ x_{e_y} & y_{e_y} & z_{e_y} & 0 \\ x_{e_z} & y_{e_z} & z_{e_z} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

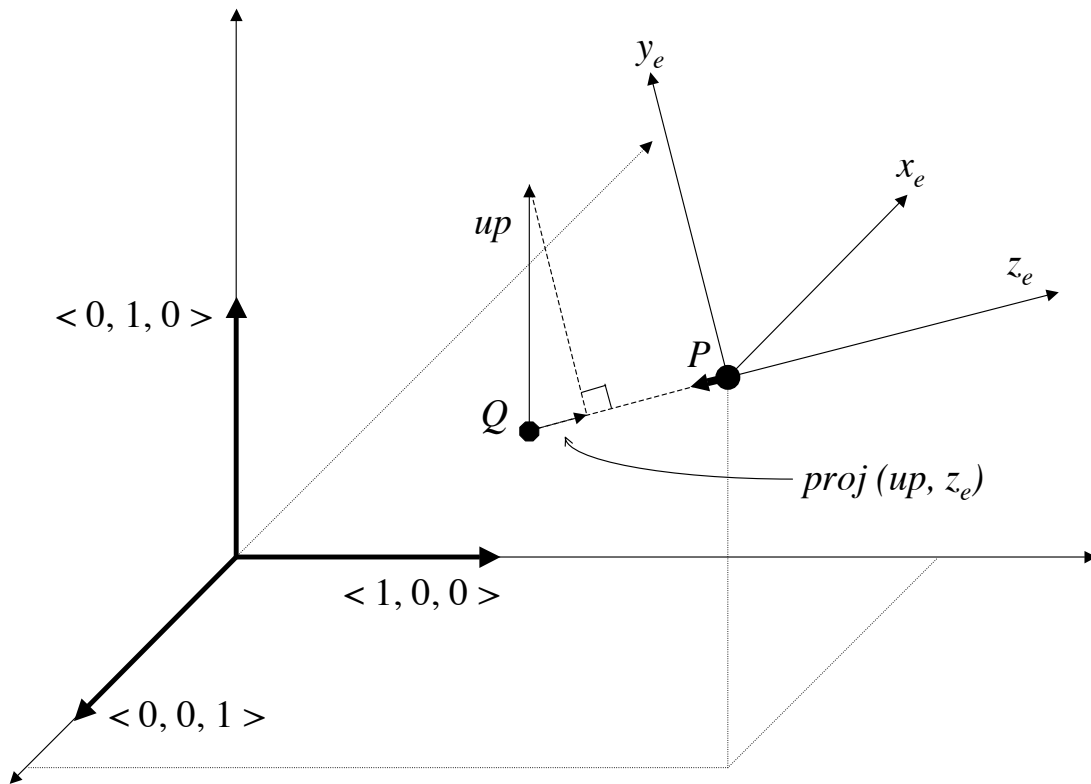


Figure 1: Setting up the camera in terms of a center of projection P looking toward the point Q with some \vec{up} vector.

- So what are \vec{x}_e , \vec{y}_e , and \vec{z}_e anyway? Easier than it seems — they can actually be derived from our look-at arguments:

$$\vec{z}_e = U_{P-Q} \quad (2)$$

$$\vec{y}_e = U_{\vec{u}\vec{p}-\text{proj}(\vec{u}\vec{p}, \vec{z}_e)} \quad (3)$$

$$\vec{x}_e = \vec{y}_e \times \vec{z}_e \quad (4)$$

Note how, since \vec{z}_e and \vec{y}_e have been normalized into unit vectors, then \vec{x}_e is automatically a unit vector.

- The matrix in (1) only rotates the coordinate system — it is still centered on the origin. Thus, we need to translate the axes to P (note how matrix composition reads from right to left):

$$\begin{bmatrix} 1 & 0 & 0 & P_x \\ 0 & 1 & 0 & P_y \\ 0 & 0 & 1 & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{e_x} & y_{e_x} & z_{e_x} & 0 \\ x_{e_y} & y_{e_y} & z_{e_y} & 0 \\ x_{e_z} & y_{e_z} & z_{e_z} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (5)$$

- Remember that (5) is the transformation that converts the *axes* to match the desired axes for the camera. However, we want the transform that converts the *objects* in 3D space according to the camera. Thus, what we really want is the *inverse* of (5) — e.g. an object that is at the origin would have coordinates $(-P_x, -P_y, -P_z)$ from the perspective of a camera that is located at world coordinates (P_x, P_y, P_z) .
- Instead of multiplying out the matrices and then inverting after, we use the matrix multiplication identity $(AB)^{-1} = B^{-1}A^{-1}$. So what are the inverses of these matrices?
- The inverse of the translation matrix is straightforward: we translate by $(-P_x, -P_y, -P_z)$ instead of (P_x, P_y, P_z) :

$$\begin{bmatrix} 1 & 0 & 0 & -P_x \\ 0 & 1 & 0 & -P_y \\ 0 & 0 & 1 & -P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6)$$

- The “axis orientation” matrix is also easy to state, but a little trickier to explain; it is the matrix’s transpose (A^T):

$$\begin{bmatrix} x_{e_x} & x_{e_y} & x_{e_z} & 0 \\ y_{e_x} & y_{e_y} & y_{e_z} & 0 \\ z_{e_x} & z_{e_y} & z_{e_z} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (7)$$

- Now, to see why this is the case, think about the characteristics of \vec{x}_e , \vec{y}_e , and \vec{z}_e . If you look at the product of the axis orientation matrix and its transpose, we get:

$$\begin{bmatrix} x_{e_x} & x_{e_y} & x_{e_z} & 0 \\ y_{e_x} & y_{e_y} & y_{e_z} & 0 \\ z_{e_x} & z_{e_y} & z_{e_z} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{e_x} & y_{e_x} & z_{e_x} & 0 \\ x_{e_y} & y_{e_y} & z_{e_y} & 0 \\ x_{e_z} & y_{e_z} & z_{e_z} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \vec{x}_e \cdot \vec{x}_e & \vec{x}_e \cdot \vec{y}_e & \vec{x}_e \cdot \vec{z}_e & 0 \\ \vec{y}_e \cdot \vec{x}_e & \vec{y}_e \cdot \vec{y}_e & \vec{y}_e \cdot \vec{z}_e & 0 \\ \vec{z}_e \cdot \vec{x}_e & \vec{z}_e \cdot \vec{y}_e & \vec{z}_e \cdot \vec{z}_e & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (8)$$

\vec{x}_e , \vec{y}_e , and \vec{z}_e are all *unit vectors* (see (4), (3), and (2)) — and so $\vec{x}_e \cdot \vec{x}_e$, $\vec{y}_e \cdot \vec{y}_e$, and $\vec{z}_e \cdot \vec{z}_e$ are all equal to one. Further, \vec{x}_e , \vec{y}_e , and \vec{z}_e are *mutually perpendicular* to each other — and thus the dot product of any two of these vectors is equal to zero. If you substitute those values into the product, note that you get the identity matrix — and thus, $A^{-1} = A^T$ for the “axis orientation” matrix!

- To wrap it all up, we now have our final *lookAt* matrix:

$$\begin{bmatrix} x_{e_x} & x_{e_y} & x_{e_z} & 0 \\ y_{e_x} & y_{e_y} & y_{e_z} & 0 \\ z_{e_x} & z_{e_y} & z_{e_z} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -P_x \\ 0 & 1 & 0 & -P_y \\ 0 & 0 & 1 & -P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (9)$$

- All that’s left is to multiply the matrices, yielding:

$$\begin{bmatrix} x_{e_x} & x_{e_y} & x_{e_z} & -(P \cdot \vec{x}_e) \\ y_{e_x} & y_{e_y} & y_{e_z} & -(P \cdot \vec{y}_e) \\ z_{e_x} & z_{e_y} & z_{e_z} & -(P \cdot \vec{z}_e) \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (10)$$

This is what `gluLookAt()` multiplies to the current matrix, of course substituting the \vec{x}_e , \vec{y}_e , and \vec{z}_e with their derivations in (4), (3), and (2), respectively. And yes, this transform is affine!

3 Putting Them Together

- Did you notice that this matrix is applied to the world, and not during projection? This is why the OpenGL code sequence for setting up the camera is:

```
glMatrixMode(GL_MODELVIEW);
glLoadIdentity();
gluLookAt(px, py, pz, qx, qy, qz, 0.0, 1.0, 0.0);
/* Rest of scene goes here. */
```

- If you put everything together, every vertex in OpenGL thus goes through these matrices on its journey from the 3D world to the screen or window:

$$[\text{viewport matrix}] [\text{projection matrix}] [\text{modelview matrix}] \quad (11)$$

- Note how we keep these three matrices distinct instead of multiplying them together. One reason is to maintain the ability to manipulate them separately — for example, if the window size changes, only the viewport and projection matrices will change, but not the modelview matrix (or at least it shouldn't).
- Another reason is further rendering algorithms such as hidden surface removal and clipping — these steps are best done at the gaps between these matrices.
- Up next — the details on these algorithms, and where they appear on the matrix train.