3D Viewing Episode 1

1 Overall Process

- Viewing is the process of mapping points in the *world coordinate system* to points in the *device coordinate system*. This consists of three general steps:

  1. Select the viewing volume.
  2. Project the points in the viewing volume.
  3. Map the projected points to the target device.

2 Viewing Volumes

- Figures 1 and 2 show how orthographic and frustum viewing volumes are defined in OpenGL, respectively. Figure 3 shows the same frustum viewing volume, but expressed with different arguments (e.g. `gluPerspective(fovy, aspect, near, far)`).
- Note that these aren't the only ways to define these volumes, nor are they the only volumes that you can do (remember fish-eye view?). But, since these are the volumes that OpenGL uses, we will study their implementation from that standpoint.
- So that's step 1. For step 2, we need to take whatever is in that viewing volume, and see how they project onto the plane defined by N in relation to the camera or viewpoint.
- Step 3 takes whatever is on the near plane and "transfers" it to the display device, such as a window (or sub-window) on the screen.

3 Projection to the Near Plane

- The approach we take is to *normalize* the camera coordinates into an intermediate space, and then convert that space into the device coordinates. This intermediate space is called the *normalized device coordinate* system (NDC).

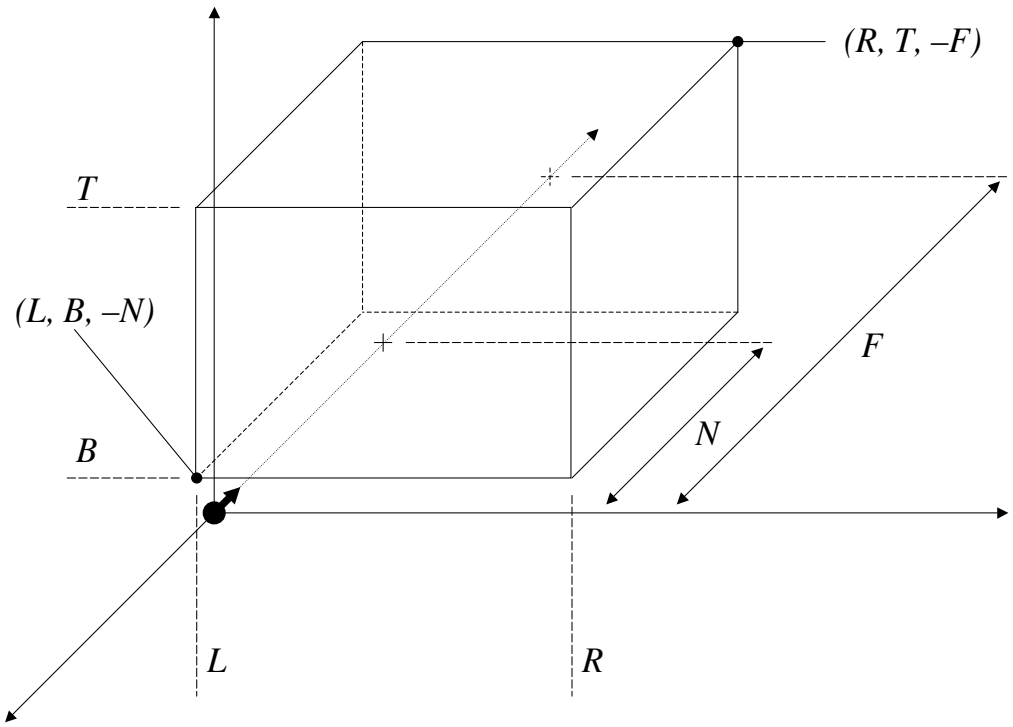


Figure 1: Orthographic viewing volume, as defined in OpenGL.

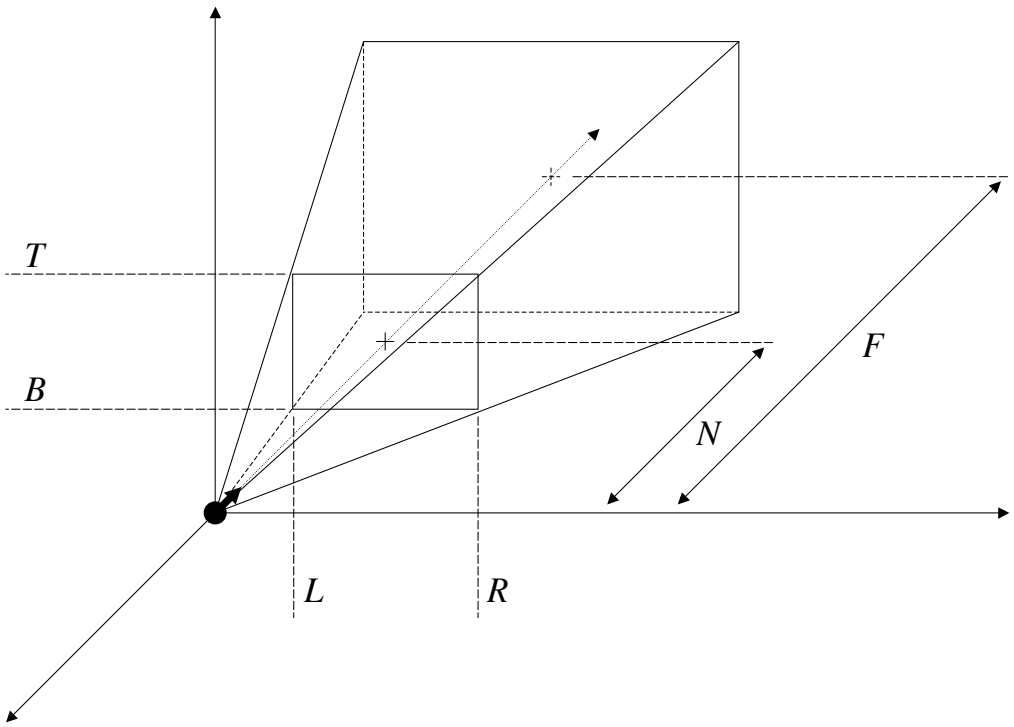


Figure 2: Frustum viewing volume, as defined in OpenGL.

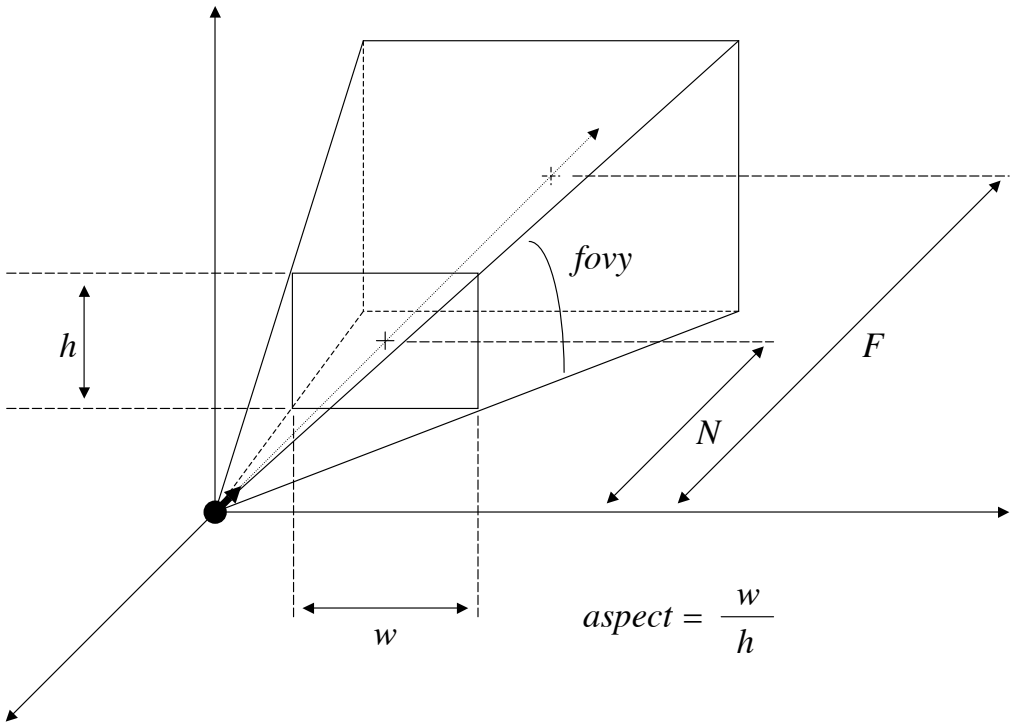


Figure 3: Frustum viewing volume, as defined in OpenGL via `gluPerspective(fovy, aspect, near, far)`.

- This normalized space converts any (x, y, z) within the viewing volume so that:

$$-1 \leq x \leq 1$$

$$-1 \leq y \leq 1$$

$$-1 \leq z \leq 1$$

- Thus, the conversion is to map $L \rightarrow -1$ and $R \rightarrow 1$ along the x axis and $T \rightarrow -1$ and $B \rightarrow 1$ along the y axis.

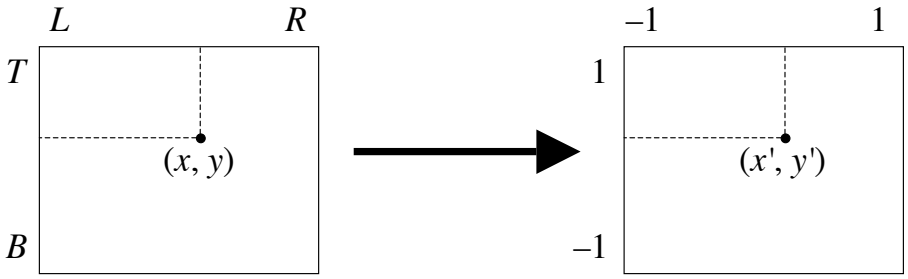
- Note however that along the z axis, the mapping is $-N \rightarrow -1$ and $-F \rightarrow 1$, since N and F are expressed as distances from the camera but our coordinate system is right-handed.

3.1 Orthogonal Projection

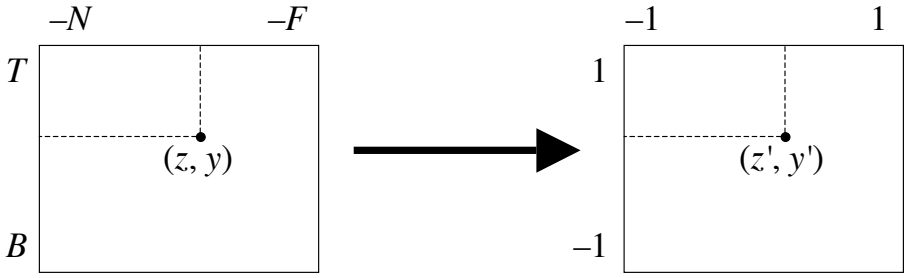
- Observe how, in an orthogonal viewing volume, (x, y) projects to exactly (x, y) on the near plane; no calculations are necessary.

- Figure 4 illustrates this visually. For our derivation, we will convert (x, y, z) in terms of camera coordinates into (x', y', z') in normalized device coordinates.

- Pause for a moment to think about it — how would you derive this?



View along the *xy* plane



View along the *yz* plane

Figure 4: Orthogonal conversion from camera to normalized device coordinates.

- OK, time's up. Note how we can do everything with ratios:

$$\frac{x-L}{R-L} = \frac{x'-(-1)}{1-(-1)} = \frac{x'+1}{2}$$

- Then we just solve for $x'\ldots$

$$x' = \frac{2(x-L)}{R-L} - 1 = \frac{2x-2L-(R-L)}{R-L}$$

$$x' = \left(\frac{2}{R-L}\right)x - \left(\frac{R+L}{R-L}\right) \tag{1}$$

- Deriving y' is analogous, leading to:

$$y' = \left(\frac{2}{T-B}\right)y - \left(\frac{T+B}{T-B}\right) \tag{2}$$

- Ditto for z', taking note of how we're using $-N$ and $-F$:

$$z' = \left(\frac{-2}{F-N}\right)z - \left(\frac{F+N}{F-N}\right) \tag{3}$$

- This is a transform — and so it can be expressed as a matrix!

$$\begin{bmatrix} \frac{2}{R-L} & 0 & 0 & -\frac{R+L}{R-L} \\ 0 & \frac{2}{T-B} & 0 & -\frac{T+B}{T-B} \\ 0 & 0 & \frac{-2}{F-N} & -\frac{F+N}{F-N} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

3.2 Perspective Projection

- For perspective projection, we have to make an extra calculation to see where any arbitrary (x, y) lands at $z = -N$. Figure 5 shows this relationship, looking down on the xz plane.

- Based on Figure 5, we can invoke the proportions between similar triangles to say:

$$\frac{x}{z} = \frac{x_e}{-N}$$

$$x_e = \frac{Nx}{-z}$$

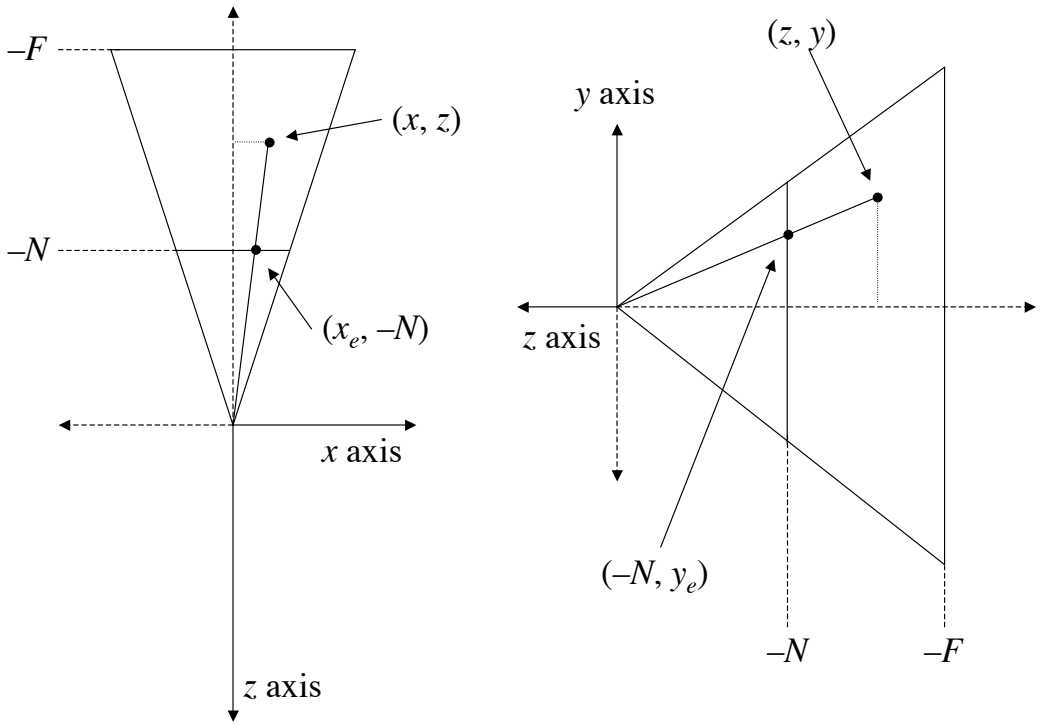


Figure 5: Perspective conversion from camera to normalized device coordinates.

- The same goes for y and y_e:

$$\frac{y}{z} = \frac{y_e}{-N}$$
$$y_e = \frac{Ny}{-z}$$

- Now we can plug x_e and y_e into (1) and (2), then substitute in terms of x and y:

$$\begin{aligned} x' &= \left(\frac{2}{R-L}\right) x_e - \left(\frac{R+L}{R-L}\right) \\ &= \left(\frac{2}{R-L}\right)\left(\frac{Nx}{-z}\right) - \left(\frac{R+L}{R-L}\right) \\ &= \frac{\left(\frac{2N}{R-L}\right) x + z\left(\frac{R+L}{R-L}\right)}{-z} \end{aligned}$$

$$\begin{aligned} y' &= \left(\frac{2}{T-B}\right) y_e - \left(\frac{T+B}{T-B}\right) \\ &= \frac{\left(\frac{2N}{T-B}\right) y + z\left(\frac{T+B}{T-B}\right)}{-z} \end{aligned}$$

- Based on these, we can partially build our projection matrix — note how the division by $-z$ is handled by the last row, based on our use of homogeneous coordinates:

$$\begin{bmatrix} \frac{2N}{R-L} & 0 & \frac{R+L}{R-L} & 0 \\ 0 & \frac{2N}{T-B} & \frac{T+B}{T-B} & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{5}$$

- But what are α and β? Based on matrix multiplication, we can say that:

$$z' = \frac{\alpha z + \beta}{-z}$$

- Since the mapping to NDC will have $z = -N$ when $z' = -1$ and $z = -F$ when $z' = 1$, then we can replace z and z' with these identities to get two equations in two variables, namely α and β:

$$-1 = \frac{\alpha(-N) + \beta}{N}$$
$$1 = \frac{\alpha(-F) + \beta}{F}$$

- Multiplying out the denominators yields:

$$-N = -\alpha N + \beta \tag{6}$$
$$-F = \alpha F - \beta \tag{7}$$

- Adding (6) and (7) eliminates β:

$$\begin{aligned} -N - F &= -\alpha N + \alpha F \\ -(F+N) &= \alpha(F-N) \end{aligned}$$

$$\alpha = -\frac{F+N}{F-N} \tag{8}$$

- Substituting (8) for α in (7) leads to:

$$\begin{aligned} -N &= \left(\frac{F+N}{F-N}\right) N + \beta \\ \beta &= -N - \left(\frac{F+N}{F-N}\right) N \\ &= \frac{-N[(F-N)+(F+N)]}{F-N} \\ &= \frac{-2NF}{F-N} \end{aligned}$$

- ... and so we have our perspective projection matrix!

$$\begin{bmatrix} \frac{2N}{R-L} & 0 & \frac{R+L}{R-L} & 0 \\ 0 & \frac{2N}{T-B} & \frac{T+B}{T-B} & 0 \\ 0 & 0 & -\frac{F+N}{F-N} & \frac{-2NF}{F-N} \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{9}$$

4 Projection to Viewport

- Projected points on the near plane now need to be converted into the display device's coordinates — the *viewport*.
- Another way to think about this is to convert from NDCs to *device coordinates* (DCs).
- For a typical window or screen with dimensions $(width, height)$, this would be a coordinate system with some origin (x_o, y_o) on the upper left corner and $(x_o + width - 1, y_o + height - 1)$ on the lower right — which is precisely what you set when you call `glViewport()`.
- Yet again, similar ratios come to our aid, based on Figure 6. Thus:

$$\frac{x' - (-1)}{1 - (-1)} = \frac{x' + 1}{2} = \frac{x_s - x_o}{width}$$

$$x_s = \frac{width(x'+1)}{2} + x_o = \left(\frac{width}{2}\right) x' + \frac{width + 2x_o}{2}$$

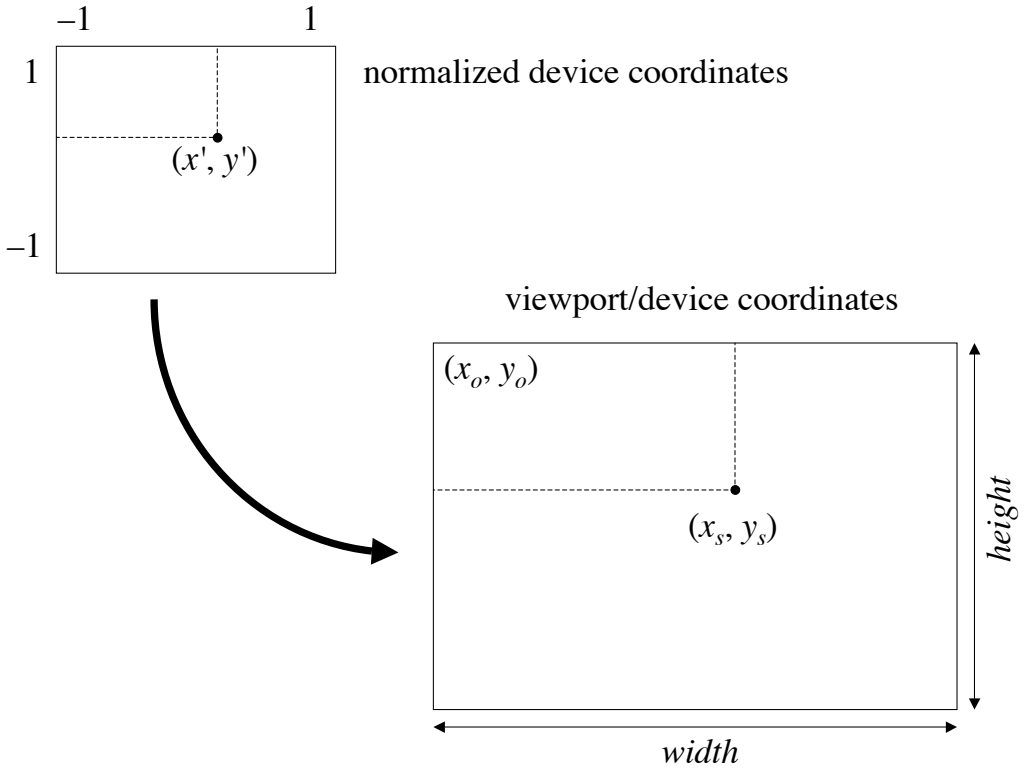


Figure 6: Conversion to viewport (device) coordinates.

- y_s is similar, except that you need to "invert" along the y axis in the end since device coordinates typically increase from top to bottom:

$$\begin{aligned} y_s &= height - \left[\left(\frac{height}{2}\right) y' + \frac{height + 2y_o}{2}\right] \\ &= -\frac{height}{2} y' + \left(height - \frac{height + 2y_o}{2}\right) \end{aligned}$$

- Of course, this is all best processed as a matrix, 3×3 this time because it's two-dimensional.

- But wait — what happened to z'? We didn't throw that away — we'll be using that later on.

- But wait again — through all of this, hasn't the camera been fixed at $(0, 0, 0)$, looking down the z-axis? The answer is yes. We haven't covered how we handle moving and orienting the camera. Stay tuned...