Algorithm Efficiency and Correctness

- At this point, we've seen a number of examples on how we can successfully develop algorithms to get certain tasks done
- We've also seen that there may be more than one algorithm to accomplish the same task, such as searching and sorting
- So what's the difference other than the algorithm itself? How would we choose one over the other? Can we make this choice *objective* or *quantitative*?

A Little Exploration

Consider a comparison between sequential and binary search — suppose we have a million records to search

- With sequential search, on average we would search half of the list — 500,000 records — before we find what we're looking for
- If the item we're looking for is the first one in the list, then we pull a single record; if it's the last one, then we end up pulling all one million records
- With binary search, we look at one record — the middle one — per "turn," then split the list in half; so, we go from a million to 500,000…250,000…125,000…62,500…31,250…15,625…7,812…3,906…1,953…976…488…244…122…61…30…15…7…3…1…that's a maximum of ***20*** "turns" to go through all million records
- For binary search, we pull a single record if the one we're looking for is in the *middle* of the list (instead of the first), and we pull the maximum of 20 if the item we're looking for requires a reduction all the way to a single-item list — note that we may find it sooner

How "Big" is an Algorithm?

- The exercise that we just went through — thinking about how many steps we would have to take in order to accomplish sequential vs. binary search — falls under the area of *algorithm analysis*
- Algorithm analysis tries to "measure" an algorithm; when it comes to the "turns" that we looked at, the number of "turns" translates into *time* — one of the major measurements for an algorithm
- Another major measure is *space* — how much memory does an algorithm need?

- An important element of algorithm analysis is that we must look at our algorithm *in general* — that is, over a wide range of instances
  - For searching, this translates to thinking about all possible combinations of lists and search targets that we may encounter when using the algorithm
- As a first cut, we note that there are *best*, *worst*, and *average* cases for an algorithm — and that these situations may vary widely

	Best Case	Worst Case	Average Case
Sequential Search	First item in the list: one "turn"	Last item in the list: one million "turns"	Middle of the list: 500,000 "turns"
Binary Search	Middle of the list: one "turn"	Reduce list to one or less: 20 "turns"	Halfway through the reduction: 10 "turns"

  - Note that the table is *specific* to a list of one million items — so still not quite general
  - Still, the table strongly suggests that binary search is the faster algorithm overall, in general

How Does an Algorithm "Grow?"

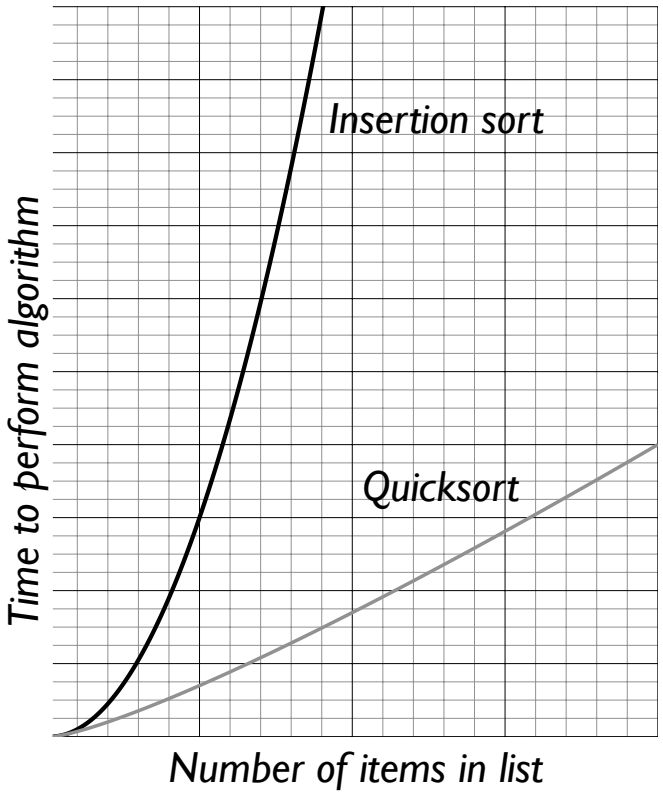
- What if, instead of supposing that we have a list of one million items, we have a list of *n* items? This truly generalizes our analysis of the algorithm…
- Note that it isn't too hard to replace "one million" with *n* in the previous table

	Best Case	Worst Case	Average Case
Sequential Search	First item in the list: one "turn"	Last item in the list: *n* "turns"	Middle of the list: *n* / 2 "turns"
Binary Search	Middle of the list: one "turn"	Reduce list to one or less: $log_2 n$ "turns"	Halfway through: $log_2 (n / 2)$ "turns"

- Let's try this exercise with insertion sort:
  - Insertion sort's best case is a list that is already sorted, since we just perform a single comparison (to the item above the "hole") for each pivot position — thus, for a list of *n* items, we would make *n* – 1 comparisons (since we start with the second item in the list)
  - At worst, the pivot always has to move to the top of the list — this would happen if the list is in reverse order — and so pivot position *i* would have to make *i* comparisons (since it is *i* slots away from the top)
  - Since we still go through each pivot position, we would make 1 comparison for pivot 1, 2 comparisons for pivot 2, etc. — the net result is 1 + 2 + 3 + … + *n* – 1, which is the summation from 1 to *n* – 1, or (following the well-known formula) (*n* – 1)*n* / 2
  - For the average case, we just figure that we move halfway to the top for each pivot position; this is just half of the worst-case summation, so we get (*n* – 1)*n* / 4
- The analysis for quicksort is much more involved, but it has been done and these are the results:
  - In the best case, we get $n \log_2 n$, with the average case coming in at approximately $2n \log_2 n$ — as with binary search, the $\log_2 n$ comes from repeatedly splitting the list in half
  - At worst, quicksort performs at around n^2 — this is when, instead of splitting the list in half, we always have the pivot at the first element (resulting in no splits at all!)

"Seeing" the Algorithm "Grow"

- If we take a pessimistic view and focus on the worst-case scenario, then we can have some kind of guarantee that, no matter what, an insertion sort will never take longer than $(n - 1)n / 2$ comparisons
- Thus, we can say that, at worst, it will take 45 comparisons to sort a list of ten items, 4,950 comparisons to sort 100 items, and *499,999,500,000* comparisons to sort one million items!
- The numbers are interesting, but there's a better way to see how the algorithm grows:



Big Theta (Θ): Efficiency by Shape

- Note how the "shape" of the graphs gives us an idea of how an algorithm fares depending on the "size" of the problem that it is solving — in the case of sorting and searching, this involves the number of items in the list
- Other algorithms may have different measures for "problem size;" for example, *make change* may vary by the number and value of coin denominations available
- In any case, computer science has a standard notation for characterizing the efficiency of an algorithm for a problem of size n — this is "big theta" or simply Θ

- Big theta notation is written as Θ(*expr*), where *expr* is a mathematical expression written in terms of n
- Further, we usually eliminate constant coefficients and addends, since these don't typically contribute much in how an algorithm grows (or deteriorates)
  - We can therefore have insertion sort being $\Theta(n^2)$
  - Quicksort is $\Theta(n \log_2 n)$ on average and $\Theta(n^2)$ at worst
  - Looking back on our search algorithms, we can then say that sequential search is $\Theta(n)$ and binary search is $\Theta(\log_2 n)$
- Comparing how *expr* changes with respect to n thus allows us to compare algorithms in a quantitative way
- Θ(*expr*) is typically read as "big theta of *expr*" or "order of *expr*" — leading to an alternative (less formal) name for the notation: "big O"

Correctness and Verification

- It's one thing to design an algorithm and *believe* that it works correctly…but is there a way to *know* that it does work correctly?
- This is another wide open area within computer science: formal verification of the correctness of an algorithm or of a computer program that claims to execute an algorithm
- The task resembles mathematical proof in many ways; in fact, there are those who might argue that there is no difference at all

- One approach to verification flows in this way:
  - Specify the initial assumptions (or *preconditions*) that are taken to be true at the start of a program (note the analogy to mathematical axioms or definitions)
  - Note that actions in an algorithm or program (assignments, conditions, loops, recursion) may gradually transform these preconditions into other states or effects
  - At a given point in an algorithm, one then makes *assertions* of what should be true at that moment; the algorithm is correct if each assertion can be shown to always hold true
- Formal verification methods have not yet seen much adoption or use; thus, *bugs* continue to abound in software, requiring extensive [manual] testing
- For the moment, many software projects use automated *unit tests* to (partially) verify correctness
  - A unit test is a program that tests another program; the program being tested performs certain functions, and the unit test makes *assertions* on what must be true after each task
  - Unit tests are only as thorough as the programmer who wrote them — they help validate certain situations but do not constitute absolute proof…still they're better than nothing